Readers' Forum

Brief discussions of previous investigations in the aerospace sciences and technical comments on papers published in the AIAA Journal *are presented in this special department. Entries must be restricted to a maximum of 1000 words, or the equivalent of one* Journal *page including formulas and figures. A discussion will be published as quickly as possible after receipt of the manuscript. Neither the AIAA nor its editors are responsible for the opinions expressed by the correspondents. Authors will be invited to reply promptly.*

The Response of Normal Shocks in Diffusers

Miklos Sajben*
McDonnell Douglas Research Laboratories
St. Louis, Missouri

IN Ref. 1, Culick and Rogers present a useful analysis of the response of normal shocks in divergent channels to periodic, low-frequency downstream perturbations. This Note complements Ref. 1 with a discussion of the time-dependent pressure distributions near the shock and extends Fig. 3 of that article (valid in the $\omega \to 0$ limit) to finite frequencies.

Figure 1 illustrates pressure distributions for (a) a stationary shock at $\bar{x}_s$ and (b) the instantaneous pressure distribution for a shock moving downstream at $\bar{x}_s + x_s'$, $x_s' > 0$. Two additional lines are also shown: the locus of pressures immediately upstream of stationary shocks $[p_u(x)]$ which depends on x via the area distribution $S(x)$ of the particular channel, and the locus of pressures immediately downstream of stationary shocks $[p_d(x)]$, also dependent on x via $S(x)$. Upstream of the shock $p_u(x)$ and line (a) coincide. Points of importance are labeled by capital letters and are used as subscripts.

For $\omega \to 0$, the pressure jumps from p_u to p_d across the shock. For the shock shown moving downstream at some finite velocity, the shock is weaker, and the instantaneous postshock pressure p_D is below the p_d curve, as shown in Fig. 1. For similar reasons, p_D behind an upstream moving shock lies above the p_d curve.

Reference 1 calculates p_D by considering the static changes of p_l and M_l associated by the shock displacement from $\bar{x}_s$ to $\bar{x}_s + x_s'$ and also includes the dynamic change in M_l caused by the motion of the shock. The pressure p_D cannot be used to calculate admittance because it exists at the time-dependent location $\bar{x}_s + x_s'$, whereas the admittance A_0 is associated with a nonmoving, constant boundary at $\bar{x}_s$. References 1 and 2 both resolve this difficulty by adding a fourth step to the calculation that consists of an extrapolation from p_D to p_E (from $\bar{x}_s + x_s'$ to $\bar{x}_s$), using the slope associated with the subsonic portion of line (a) at $\bar{x}_s$. Thus the pressure perturbation p_e' is associated with the fixed location $\bar{x}_s$ and defined in terms of Fig. 1 as $p_e' = p_E - p_A$.

The trajectory of point D on the $p(x)$ plane during harmonic oscillations of the shock can be easily determined in the manner briefly outlined below. The results of the first three calculation steps can be written in the form:

$$p_D - p_A = B_v v_s' + B_x x_s' \tag{1}$$

where the B's are known, constant coefficients. Since for the static case ($v_s' = 0$) p_D lies on $p_d(x)$, the quantity B_x is equal to the slope of the $p_d(x)$ curve at point A. Next, let $x_s' = X \cos \omega t$

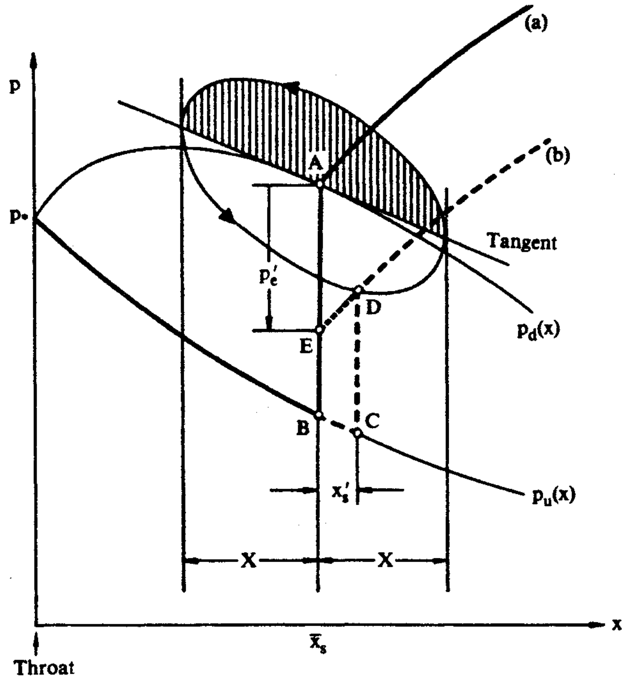

Fig. 1 Pressure distributions for (a) stationary shock and (b) downstream moving shock. Vertical shading highlights the elliptical locus for the pressure immediately downstream of a shock oscillating sinusoidally.

(x_s', X both real) from which:

$$v_s' = \frac{dx_s'}{dt} = X\omega\sqrt{1 - (x_s'/X)^2} \tag{2}$$

Eliminating v_s' from Eq. (1), using Eq. (2), and rearranging yields:

$$\left[\frac{p_D - (p_A + B_x x_s')}{B_v X \omega}\right]^2 + \left[\frac{x_s'}{X}\right]^2 = 1 \tag{3}$$

The quantity in parentheses in the first term defines the tangent line to $p_d(x)$ at point A, while the numerator of the same term is the vertical distance of p_D from this line. The form of Eq. (3) indicates that the p_D trajectory is an ellipse, plotted in an oblique coordinate system whose origin is point A and whose axes are defined by the tangent to $p_d(x)$ and the vertical direction (Fig. 1).

Since the vertical axis of the ellipse is proportional to frequency, the representation clearly illustrates frequency effects on the relative amplitude and phase of the pressure and shock-displacement fluctuations. For $\omega \to 0$ the ellipse collapses onto the tangent line, and the pressure and displacement are out of phase by 180 deg.

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*Staff Manager-McDonnell Douglas Fellow. Associate Fellow AIAA.

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Comment on "Derivation and Significance of Second-Order Modal Design Sensitivities"

Alexander H. Flax*

Institute for Defense Analyses, Alexandria, Virginia

BRANDON'S derivation[1] of the "second-order modal design sensitivities" leads to results identical to the first- and second-order terms obtained from Rayleigh's classical perturbation theory[2] for variations in stiffness, but the second-order terms are in error for variations in mass. Rayleigh's methods are well known in the literature.[3-4] His formulation is general and valid whether the eigenvalue problem is in matrix, differential, or integral equation form.

Brandon's perturbation eigenvalue problem in matrix form is

$$(\lambda A + B)\{x\} = \epsilon Q\{x\} \tag{1}$$

where A and B are the mass and stiffness matrices, respectively. $\{x\}$ is the (column) eigenvector, ϵ the parameter representing the magnitude of the perturbations, and λ the eigenvalue ($= -\omega^2$, where ω is a natural frequency). Q is the matrix of perturbation terms. However, the combination of perturbations in A and B in this manner is inadmissible if perturbations of higher order than the first are to be calculated. This is because variations in A and variations in λ are multiplied, leading to terms in ϵ of second and higher degrees not found from Eq. (1). The perturbation equation which must be used is

$$[\lambda(A+\alpha) + (B+\epsilon\beta)]\{x\} = 0 \tag{2}$$

where α is the matrix of perturbation terms in mass and β the matrix of perturbation terms in stiffness.

Rayleigh's formulation[2] for the second order perturbation in the nth eigenvalue $\lambda_n^{(2)}$ gives

$$\lambda_n^{(2)} = -\frac{B_{n\eta} + \epsilon\beta_{n\eta}}{A_{nn} + \epsilon\alpha_{nn}} + \epsilon^2 \sum_{m\neq\eta} \frac{(\beta_{nm} + \lambda_{\eta 0}\alpha_{nm})^2}{A_{nn}A_{mm}(\lambda_{n0} - \lambda_{m0})} \tag{3}$$

Here, if the jth eigenvalue and eigenvector of the unperturbed problem are λ_{j0} and $\{\phi_j\}$, respectively,

$$D_{nm} = \{\phi_n\}^T D\{\phi_m\} \tag{4}$$

and

$$\lambda_{n0} = -(B_{nn}/A_{nn}) \tag{5}$$

where the matrix D may be taken as A, α, B, or β and the superscript T denotes the transpose.

The first term on the right of Eq. (3) is the Rayleigh quotient and is not yet limited to terms up to the second degree in a power series in ϵ. To achieve this requires expansion of the denominator in a power series leading to

$$\frac{B_{nn} + \epsilon\beta_{nn}}{A_{nn} + \epsilon\alpha_{nn}} = \frac{B_{nn} + \epsilon\beta_{nn}}{A_{nn}}\left[1 - \frac{\epsilon\alpha_{nn}}{A_{nn}} + \frac{\epsilon^2\alpha_{nn}^2}{A_{nn}^2} + \cdots\right] \tag{6}$$

To second order in ϵ, this gives

$$\frac{B_{nn} + \epsilon\beta_{nn}}{A_{nn} + \epsilon\alpha_{nn}} \approx \frac{B_{nn}}{A_{nn}} + \epsilon\left(\frac{\beta_{nn}}{A_{nn}} - \frac{B_{nn}}{A_{nn}^2}\alpha_{nn}\right) + \epsilon^2\left(\frac{B_{nn}}{A_{nn}}\frac{\alpha_{nn}^2}{A_{nn}^2} - \frac{\beta_{nn}\alpha_{nn}}{A_{nn}^2}\right) \tag{7}$$

Using Eq. (5) for λ_{n0} and normalizing the unperturbed eigenvectors so that $A_{jj} = 1$ gives

$$\lambda_n^{(2)} = \lambda_{n0} - \epsilon(\beta_{nn} + \lambda_{n0}\alpha_{nn}) + \epsilon^2\alpha_{nn}(\beta_{nn} + \lambda_{n0}\alpha_{nn}) + \epsilon^2 \sum_{m\neq n} \frac{(\beta_{nm} + \lambda_{n0}\alpha_{nm})^2}{(\lambda_{n0} - \lambda_{m0})} \tag{8}$$

where $\lambda_n^{(2)}$ is the perturbation expansion up to terms of the second order. Brandon's expression for the terms in ϵ^2 in the perturbation power series consists only of the last term in this equation and omits the other terms in ϵ^2 as would be expected from the approach taken in his derivation.

The expansion of the denominator of the left-hand side of Eq. (6) can have a powerful effect on the accuracy of higher order perturbation calculations. A particularly fortuitous case involving linearly tapered beams in which the eigenvalues were predicted to high accuracy by second-order perturbation theory was shown in Ref. 6 to result in large measure from the fact that the Rayleigh quotient was given exactly by a polynomial of the second degree in the perturbation parameter so that no series expansion of the denominator of the quotient was necessary.

In general, perturbation theory to the first few orders of the perturbation parameter deteriorates in accuracy as higher vibration modes are considered, especially when there are substantial perturbations in the mass terms, (e.g., the problem of a cantilever beam with eccentric tip mass[7]). Perturbation theory and its strengths and weaknesses have been extensively explored, but the available results seem to be often overlooked, as Brandon has pointed out, by investigators seeking design sensitivity factors for structural optimization analyses.

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*President Emeritus. Honorary Fellow AIAA.